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**Algebraic Topology:
An Intuitive Approach**

Hajime Sato

**Translated by
Kiki Hudson**



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ISŌ KIKI
(ALGEBRAIC TOPOLOGY)

by Hajime Sato
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ABSTRACT. This book develops an introduction to algebraic topology mainly through simple examples built on cell complexes. The topics covered include homeomorphisms, homotopy equivalences, the torus, the Möbius strip, closed surfaces, the Klein bottle, cell complexes, fundamental groups, homotopy groups, homology groups, cohomology groups, fiber bundles, vector bundles, spectral sequences, and characteristic classes.

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Preface

Topology forms a branch of geometry emphasizing connectedness as the most fundamental aspect of a geometrical object. In topology, therefore, one ignores virtually all geometrical traits other than connectedness, such as any form of change in a geometrical object that stretching or shrinking might cause. Classification in topology is a crude tool, but one that never fails to determine if a geometrical object is connected or not. If a geometrical object is connected then we investigate to what degree it is connected. Just as the state of connectedness characterizes the essence of many phenomena we encounter in our daily lives, it is often necessary to describe to what extent a certain object is connected or separated. Thus the terms one employs in topology are increasingly becoming important and useful in other branches of mathematics as well as in various fields in the natural sciences.

There are numerous algebraic topology books and many of them are excellent; yet we have dared to add another book on this subject. The single most difficult thing one faces when one begins to learn a new branch of mathematics is to get a feel for the mathematical sense of this subject. To somebody who has mastered the subject this essential common sense should be as familiar as the air around him. It takes a long time for a beginner to get to this point. The purpose of this book is to help an aspiring first-time reader acquire this topological atmosphere in a short period of time.

I believe that the most efficient way to fulfill this purpose is to investigate simple but meaningful examples in some concrete terms. It is important that the reader grasp a mathematical object with his or her own hands. By touching it one can feel its physical quality and then keep this as one's own. This book is a simple manual that the reader can follow, and in fact the reader who follows our instructions step by step will end up with a real working model of algebraic topology.

In order to pursue this objective we have therefore sacrificed generality and limited the objects of our discussion to the simplest but most essential cases. We did not try to expand the theory to its fullest extent to make our book an encyclopedic reference; instead, we use the easiest possible examples to help the reader see the backbone of our discussion.

We will be greatly pleased if the reader enjoys reading our book while acquiring several essential methods or approaches to discuss algebraic topology. We must await the reaction of the reader to see if our plan will succeed. We will appreciate it if the reader gives us any feedback (criticisms and comments)‘.

The basic framework of the book comes from the seminar notes “Practical Topology for Physicists” given by Akihiro Tsuchiya and compiled by Yasuhiko Yamada at the University of Nagoya in 1986. I am deeply indebted to Mr. Tsuchiya for permitting me to use his seminar notes as well as for giving me much useful advice throughout every stage of the writing. My thanks also go to Tadayoshi Mizutani, Tetsuya Ozawa, Yoshinori Machida, and Shigeo Ichiraku, who not only read the entire manuscript carefully, finding many mistakes, but also suggested various ways to improve the final product. Last but not least, I would like to thank the editors at Iwanami Shoten.

Hajime Sato
July 1996

‘See Preface to the English Translation

Preface to the English Translation

It is a great pleasure to me that the American Mathematical Society chose to publish my book “Algebraic Topology: An Intuitive Approach” in their translation series.

Since the publication of the original version of this book in 1996, several of my friends (including the translator) have complained that the gap between my claim that *no previous knowledge of mathematics is required*. . . and the actual contents of the book is too big. So I have provided the reader who has no knowledge of sets, topology, groups, *etc.* with a basic minimal list of definitions and results that may prove useful, together with readable references. This is in the Appendix at the end of the book. This does not really change my original view that the book is readable for anybody who wishes to find out about algebraic topology. I think that technical terms help both the reader and the author organize their thoughts, but they will not do much good unless both the reader and the author have “good vibes” about the subject. I have also used the book for my topology seminar (for seniors) and came to see that the reading got a little rough toward the end of the book. This is all right too, since it simply shows that good vibes alone cannot conquer everything; however, I have modified some of those troublesome spots, filling in missing links and so on.

I am grateful to the translator, Kiki Hudson, for conveying my writing style and philosophy as faithfully as possible in her translation. We discussed all the changes verbally, and consequently she had to do more writing than translating. This is especially so with the Appendix. I would also like to thank Martin Guest for valuable suggestions, Yoshinori Machida for spotting numerous typos, and the AMS editors for presenting the book in splendid style.

Hajime Sato
September 1998

Objectives

As I stated in the Preface, in topology we investigate one aspect of geometrical objects almost exclusively of the others: that is, whether a given geometrical object is connected or not connected. We classify objects according to the nature of their connectedness. One focuses on the connectivity, ignoring changes caused by stretching or shrinking.

One can measure the length of a geometrical object in meters and the weight in kilograms. How do we measure the extent to which a geometrical object is connected? Can we develop a system with suitable units and numbered scales?

For example, we can use the number of holes in a geometrical object. But then what is a hole and how do we count the number of holes? In this book, you will find a mathematical interpretation of these concepts, termed “homotopy groups”, “homology groups”, and “cohomology groups”. These are some of the major concerns in algebraic topology. We actually go beyond counting the number of holes and develop “characteristic classes” to describe how a geometrical object bends globally. Intuitively the “ i -th homotopy group” describes the “ i -dimensional *round holes*” and “ i -th homology group” reveals the number of “ i -dimensional rooms” in a geometrical object.

In the problem described above, which may appear to be too slippery to grasp, it would be nice if the reader would come to understand and appreciate how contemporary mathematics has constructed the theory of algebraic topology, translating geometrical concepts into algebraic terms. It has managed to express these problems cleanly and algebraically in group-theoretical terms (involving almost only the additive group of integers or cyclic groups of integers modulo prime numbers). I want the reader to spend a few minutes before beginning the book imagining the problem of classifying geometrical objects only with a yardstick that measures their connectedness. Then after finishing the book the reader should compare its contents with this original concept. If the concept and reality are far apart you will have

opened a door to a brave new world, and if they are rather close your mathematical intuition will have proved to be excellent (and you will continue to go on the right track with conviction).

If you already have any familiarity with algebraic topology, you might rightly guess from the table of contents that the following are the key words in the book:

homeomorphisms, homotopy equivalences, torus, Möbius strip, closed surfaces, Klein bottle, cell complexes, fundamental groups, homotopy groups, homology groups, cohomology groups, fiber bundles, vector bundles, spectral sequences, characteristic classes, etc.

If you have seen some (or all) of these words somewhere before and they have vaguely interested you, then you will find upon finishing the book that they are not difficult at all but that they form some of the basic concepts in contemporary mathematics. If you have had nothing to do with them so far, I hope that the strange sound they make intrigues you enough to start the book.

Topology has developed (perhaps unintentionally) on the strength of several attractive geometrical figures which serve as characteristic examples for the theory. This pattern may not be unique in topology; we may see it repeated in other branches of mathematics and possibly in every other academic discipline.

I emphasize again that the purpose of this book is to familiarize the reader with the way to think about algebraic topology. I use the axiomatic approach to introduce homology and cohomology theories, and will later construct concrete examples such as simplicial homology groups, as I feel that this order might work better to sharpen the reader's intuitive understanding.

Needless to say, algebraic topology evolved from general topology (the theory of topological spaces). If you have already studied general topology (especially its geometrical aspects), for instance if you have read Chapters from I to XI in *Topology* of James Dugundji², you will be ideally prepared; however, I have tried to keep my explanation basically intuitive so that even readers with no previous knowledge of general topology will be able to follow the book.

The reader might feel a need for the theory of groups, but essentially all you need in order to read this book is to understand the following two concepts:

² *Topology* by James Dugundji, William C. Brown. 1989

(1) The addition or subtraction of two integers gives another integer (we say that the set \mathbb{Z} of the integers is an additive group).

(2) In certain situations, we regard two integers which differ by a fixed prime number p to be equal (we say that we consider integers mod p). We write \mathbb{Z}_p for the set of the integers mod p . The addition and subtraction of integers carry over to those operations mod p (we say that mod p is a cyclic group of order p).

The only talent this book demands of the reader is a flexible and resilient mind.

LIST OF SYMBOLS

Symbol	Meaning	Page
$f_0 \simeq f_1$	homotopic	3
$[X, Y]$	homotopy set	4
$X \simeq Y$	X and Y have the same homotopy type	4
D^n	n-dimensional ball	9
S^{n-1}	(n - 1)-dimensional sphere	9
\mathbf{I}	closed unit interval $[0, 1]$	10
$P^n(\mathbb{R})$	n-dimensional real projective plane	11
e^i	(open) i-cell	13
\bar{e}^i	closed i-cell	13
$\pi_n(X, x_0)$	n-th homotopy group of X	25
$\pi_n(X)$	n-th homotopy group of X	25
$h_p(X)$	p-th homology group of X	31
$h_*(X)$	direct sum $\sum_{p=0}^{\infty} h_p(X)$ of $h_p(X)$	31
pt	singleton set	32
$H_p(X; G)$	$h_p(X)$ for $ho(X) \cong G$	32
$H_*(X; G)$	direct sum $\sum_{p=0}^{\infty}$ of $H_p(X; G)$	32
CA	cone over A	36
$\tilde{h}_*(X)$	reduced homology of group X	38
c	chain complex	45
$Z_p(C)$	group of p-cycles	45
$B_p(C)$	group of p-boundaries	45
$\sigma^j \prec \sigma^n$	simplex σ^j belongs to the boundary of σ^n (σ^j is a face of σ^n that is different from σ^n)	48
$C_q(\mathcal{S}; \mathbb{Z})$	q-th chain group of \mathcal{S} over \mathbb{Z}	52
$H_q(\mathcal{S}; \mathbb{Z})$	q-th homology group of \mathcal{S} over \mathbb{Z}	53
$P^n(\mathbb{C})$	n-dimensional complex projective space	56
$h^p(X)$	p-th cohomology group of X	59
s	simplicial complex	49

Symbol	Meaning	Page
$h^*(X)$	direct sum $\sum_{p=0}^{\infty} h^p(X)$ of $h^p(X)$	59
δ^p, δ	coboundary homomorphism	60
$C^q(\mathcal{S}; G)$	q-th cohomology chain of \mathcal{S} over G	60
$C^*(\mathcal{S}; G)$	cochain complex of \mathcal{S} over G	61
$H^q(\mathcal{S}; G)$	q-th cohomology group of \mathcal{S} over G	61
$Z^q(\mathcal{S}; G)$	group of q-cochains of \mathcal{S} over G	61
$B^q(\mathcal{S}; G)$	group of q-coboundaries of \mathcal{S} over G	61
$G_1 \otimes G_2$	tensor product	
$\text{Hom}(G_1, G_2)$	abelian group of homomorphisms from G_1 to G_2	66
$\text{Tor}(G_1, G_2)$	torsion	66
$\text{Ext}(G_1, G_2)$	abelian group of the extensions of G_2 by G_1	67
\times	cross product	68
Δ	diagonal map	69
\cup	cup product	69
(E, π, B, F)	fiber bundle	74
$F \rightarrow E \xrightarrow{\pi} B$	fiber bundle	74
E	total space	74
B	base space	74
F	fiber	74
π	projection	74
$G^{\mathbb{R}}(m, n)$	real Grassmannian manifold	80
$G^{\mathbb{C}}(m, \mathfrak{n})$	complex Grassmannian manifold	80
$BO(n)$	classifying space of real n-vector bundles	83
$BU(n)$	classifying space of complex n-vector bundles	83
$\text{Lk}(\sigma, \mathcal{S})$	link complex of σ in \mathcal{S}	106

CHAPTER 1

Homeomorphisms and Homotopy Equivalences

Throughout this book a map means a *continuous map*.⁷

In topology we essentially discuss the connectedness of geometrical objects called topological spaces; however, strictly speaking, we consider topological spaces and two types of continuous maps between them, which are called “homeomorphisms” and “homotopy equivalences” respectively. We might classify topological spaces up to homeomorphism, or we might do so up to homotopy equivalence. Our choice depends on how strong we want our classification to be. The classification according to homotopy equivalences is weaker (there are many spaces not “homeomorphic” to each other that are of the same “homotopy type”), but it is the one that plays the more important role in algebraic topology, because geometrical properties of homotopy equivalences translate themselves most successfully into modern algebra.

The classification of the capital letters A, B, C, . . . , Z by homeomorphisms results in the following nine classes (this also depends on the choice of font, and here we use the sans-serif style; for example we write I and not l).

$$\{A, R\}, \{B\}, \{C, G, I, J, L, M, N, S, U, V, W, Z\}, \\ \{D, O\}, \{E, F, T, Y\}, \{H, K\}, \{P\}, \{Q\}, \{X\}.$$

The letters in any one of these classes are homeomorphic but no two belonging to distinct classes are.

On the other hand, homotopy classification breaks the alphabet into three distinct classes according to their “homotopy types”:

$$\{A, R, D, O, P\}, \{B, Q\}, \\ \{C, I, L, M, N, S, U, V, W, Z, F, J, T, Y, G, H, K, X\}.$$

⁷See the Appendix for the definition

Two letters have the same homotopy type if and only if they belong to the same class.

We count the number of holes in each letter in the set containing the letter A as one, that of each letter in the set containing B as two, and that of each letter in the last set as zero. Have the above simple examples led you to guess the definitions of homeomorphisms and homotopy equivalences?

1.1. Homeomorphisms

DEFINITION 1.1. We say that topological spaces X and Y are *homeomorphic* if there exist continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that the composites $g \circ f$ and $f \circ g$ are the identity maps of X and Y respectively; in short, $g \circ f = id$ and $f \circ g = id$, where id denotes the identity map. In this case f is a *homeomorphism* from X to Y and g is a homeomorphism of Y to X .

The fact that $g \circ f$ is the identity map implies that f is an injection and g is a surjection. Similarly the fact that $f \circ g$ is the identity map implies that f is surjective and g is injective. Altogether it follows that both f and g are continuous bijective (1-1 onto) maps.

SAMPLE PROBLEM 1.2. Consider the letters M and N. Think of them as topological spaces and construct homeomorphisms $f : M \rightarrow N$ and $g : N \rightarrow M$.

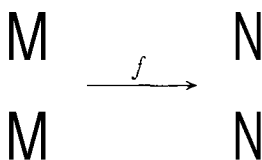


FIGURE 1.1

SOLUTION. Let f be a map which sends the left half A of the M onto the left vertical line plus the center diagonal A of the N without changing anything, while straightening the right half A of M and sending it onto the right vertical line I of N (see Figure 1.1). We want g to transfer the left vertical line and the center diagonal line of N onto the left half A of M, and to bend the right vertical line of N and map it onto the right half A of M. Then we get $g \circ f = id$ and $f \circ g = id$.

SAMPLE PROBLEM 1.3. Show that the topological spaces X and I are not homeomorphic.

SOLUTION. Suppose there existed a homeomorphism $f : X \rightarrow I$. For any point x_0 in X , the definition of a homeomorphism insures that the map $f|_{(X-x_0)}$ which is the restriction of f to the space X minus the point x_0 is a homeomorphism of $X - x_0$ onto $I - f(x_0)$. Take, in particular, the crossing point of X as x_0 . Then $X - x_0$ consists of four disjoint line segments (each being half open, having one open end and one closed end), and $I - f(x_0)$ consists of two disjoint line segments (each of which is half open). These two spaces are not homeomorphic.

The basic stance of topology is to regard all spaces homeomorphic to each other as identical.

1.2. Homotopy equivalences

In order to define homotopy equivalences we must first say when two maps are homotopic.

DEFINITION 1.4. Two maps from a topological space X to a topological space Y ,

$$f_i : X \rightarrow Y \quad (i = 0, 1),$$

are *homotopic* if there exists a family of continuous maps

$$f_t : X \rightarrow Y \quad (t \in [0, 1]),$$

varying continuously from f_0 to f_1 . We indicate this situation by $f_0 \simeq f_1$ and say that f_t ($t \in [0, 1]$) is a *homotopy* between them.

EXAMPLE 1.5. We consider two maps f_0 and f_1 from the letter X to the letter Y : f_0 sends every point of X to the crossing point of Y , and f_1 maps the upper vee v of X onto the upper vee v of Y and the lower wedge A of X onto the lower vertical I of Y by closing A like a tweezer. Then f_0 and f_1 are homotopic because we can define f_t , $t \in [0, 1]$, to be the map sending each point x of X to the point obtained by shrinking $f_1(x)$ by t from the center crossing of Y .

EXAMPLE 1.6. Take the letter O . Let f_0 be the map of O into itself which sends every point to the apex of the O and let f_1 be the identity map. Then f_0 and f_1 are not homotopic.

This fact is intuitively obvious (we can never change the identity map of O to a constant map through continuous maps: we cannot shrink the letter O to a point without breaking it). A precise proof,

however, depends on homology theory, and we will see it in Example 4.9.

Suppose we look at the set S of the maps from a topological space X to a topological space Y . The following properties are easy to check.

1. A map $f : X \rightarrow Y$ is homotopic to itself.
2. If f is homotopic to $g : X \rightarrow Y$ then g is homotopic to f .
3. If f is homotopic to g and g is homotopic to $h : X \rightarrow Y$ then f is homotopic to h .

Therefore the relation of being homotopic is an equivalence relation on S that breaks S into equivalence classes called *homotopy classes*. We denote by

$$[X, Y]$$

the set of the homotopy classes of maps from X to Y , which we call the *homotopy set* of X to Y . In other words, we regard all homotopic maps from X to Y as identical and place them in the same homotopy class. Therefore, even if a homotopy class has a large number of maps, we need to look at only one of them. This is an algebraic simplification.

EXAMPLE 1.7. Consider the letters X , Y and O . We will discuss the following result in Chapter Three:

$$[X, Y] \cong \text{one point}, \quad [O, O] \cong \mathbb{Z} \text{ (the set of the integers).}$$

DEFINITION 1.8. Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is a *homotopy equivalence* of X and Y if for some map $g : Y \rightarrow X$, the composites $g \circ f : X \rightarrow X$ and $f \circ g : Y \rightarrow Y$ are homotopic to the identity map of X and Y respectively.

We say that X and Y have the same *homotopy type* if there exists a homotopy equivalence between them.

In general a homotopy equivalence is neither injective nor surjective. We write $X \simeq Y$ when X and Y have the same homotopy type. We are using the same symbol for homotopic maps, but this should not cause any confusion here since both sides are topological spaces.

PROBLEM. Show that the map $f_0 : X \rightarrow Y$ from the letter X to the letter Y in Example 1.5 is a homotopy equivalence (Hint: for a suitable $g : Y \rightarrow X$ construct a homotopy between $g \circ f_0$ and the identity map as well as a homotopy between $f_0 \circ g$ and the identity map).

From the definition we see that two topological spaces that are homeomorphic have the same homotopy type; therefore, homotopy equivalences are a looser (less strict) way of classifying topological spaces.

We have so far used only letters of the alphabet. These are one-dimensional geometrical objects (topological spaces) consisting of lines and curves; however, the definitions of homeomorphisms and homotopy equivalences carry over to geometrical objects of dimensions two or higher, including of course three-dimensional spaces.

EXAMPLE 1.9. A doughnut is homeomorphic to a coffee cup with a handle, and has the same homotopy type as the letter 0.

In later chapters we will study homology groups and cohomology groups (of topological spaces). They each offer the identical information for spaces of the same homotopy type. We will introduce other tools such as characteristic classes to determine if the given spaces are homeomorphic.

1.3. Topological pairs

In topology we frequently consider a pair of topological spaces (X, A) rather than a single space X . Passing from single spaces to pairs of spaces as objects of study was a great breakthrough in algebraic topology in the past.

By a *topological pair* (X, A) we mean a topological space X and a subspace A of X .

Given two pairs (X, A) and (Y, B) , by a map of pairs $f : (X, A) \rightarrow (Y, B)$ we mean a map $f : X \rightarrow Y$ such that

$$f(A) \subset B.$$

The concept of homeomorphisms for topological pairs parallels the case for single spaces; namely, two pairs (X, A) and (Y, B) are homeomorphic if we can find maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that the composites $g \circ f : X \rightarrow X$ and $f \circ g : Y \rightarrow Y$ are the identity maps of X and Y respectively. The restrictions $f|_A : A \rightarrow B$ and $g|_B : B \rightarrow A$ are both homeomorphisms.

EXAMPLE 1.10. In the (x, y, z) -space \mathbb{R}^3 , splice the ends of a string to make a simple loop A . Make a loop B in \mathbb{R}^3 by tying a knot in the string before splicing its ends (see Fig. 1.2).

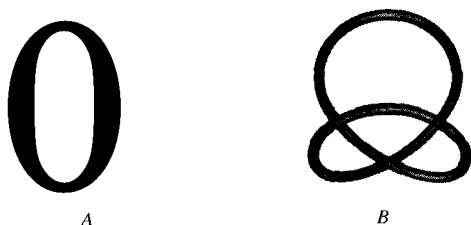


FIGURE 1.2. Knots

We can show that such pairs (\mathbb{R}^3, A) and (\mathbb{R}^3, B) are not homeomorphic to each other (later we will compute the “fundamental groups of the complements of A and B ” using “homotopy theory”).

We say that two continuous maps of pairs $f_i : (X, A) \rightarrow (Y, B)$, $i = 0, 1$, are homotopic if there exists a family of continuous maps of pairs

$$f_t : (X, A) \rightarrow (Y, B), \quad t \in [0, 1],$$

varying continuously from f_0 to f_1 .

We partition the continuous maps from a pair (X, A) to another pair (Y, B) into homotopy classes; that is, we look at the set denoted by

$$[(X, A), (Y, B)]$$

in which each element is a homotopy class consisting of all homotopic maps from (X, A) to (Y, B) . We say that $[(X, A), (Y, B)]$ is the homotopy set of maps from (X, A) to (Y, B) . In particular, if $A = B = \emptyset$ we write X and Y in place of (X, \emptyset) and (Y, \emptyset) . Then we have

$$[X, Y] = [(X, \emptyset), (Y, \emptyset)].$$

as the right-hand side of the equality is the homotopy set in which an element is a set of homotopic maps from X to Y .

We investigate detailed features of homotopy sets in Chapter Three.

Summary

1.1 A map from one topological space to another is a homeomorphism if it has an inverse map. Two topological spaces are homeomorphic if there exists a homeomorphism between them.

1.2 A map from one topological space to another topological space is a homotopy equivalence if it has an “inverse map” in the homotopy

sense. Two topological spaces have the same homotopy type if there exists a homotopy equivalence between them.

1.3 The same ideas carry over to homeomorphisms, homotopy equivalences and homotopy types for maps of topological pairs.

Exercises

1.1 Show that the letters W and Z are homeomorphic.

1.2 Show that the letters P and R have the same homotopy type.

1.3 The upper portion Δ of the letter A is a subspace of A and the upper portion D of R is a subspace of R. Show that the pairs (A, Δ) and (R, D) are homeomorphic.

CHAPTER 2

Topological Spaces and Cell Complexes

There is a large selection of geometrical objects around us, ranging from basic ones such as line segments and disks to fuzzy ones whose boundaries are blurry. We must state precisely which geometrical objects are subjects of our investigation in this book. We must be able to determine if a geometrical object is connected or separated. In other words, we only consider those objects on which we can impose the concept of continuity, and we will call them topological spaces. There is a wide variety of topological spaces, among which the most basic are (solid) balls, also referred to as disks or cells. The boundary surface of a ball is a sphere. The dimensions of cells we study do not stop after one, two and three, but run up to n in general. We construct a topological space called a cell complex by splicing together finitely many cells of suitable dimensions. In this chapter we explain how to build various topological spaces and cell complexes. In the ensuing chapters we will deal with cell complexes only, unless otherwise stated.

2.1. Basic spaces

For a natural number $n \geq 1$, we define the n -dimensional ball (or n -ball) D^n by

$$D^n = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 \leq 1 \right\},$$

and the $(n - 1)$ -dimensional sphere (or $(n - 1)$ -sphere) by

$$S^{n-1} = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 = 1 \right\}.$$

The 0-sphere S^0 consists of two points $\{\pm 1\}$. We make the convention that the 0-disk D^0 is a one-point space. The boundary ∂D^n of the